

Concavity of Perelman's \mathcal{W} -functional over the space of Kähler potentials

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Abstract

In this short note we observe that the concavity of Perelman's \mathcal{W} -functional over a neighborhood of a Kähler-Ricci soliton inside the space of Kähler potentials is a direct consequence of author's solution of the variational stability problem for Kähler-Ricci solitons. Independently, we provide a rather simple proof of this fact based on some elementary formulas obtained in our previous work.

Let (X, J, ω) be a compact Kähler-Ricci soliton and let $\omega_t := \omega + i\partial\bar{\partial}\varphi_t$ be a family of Kähler metrics with $\varphi_0 = 0$. We denote by f_t the unique function such that $i\partial\bar{\partial}f_t = \omega_t - \text{Ric}(\omega_t)$ and $\int_X e^{-f_t} \omega_t^n = n!$. We consider the Riemannian metric $g_t := -\omega_t J$ and the positive volume form $\Omega_t := e^{-f_t} \omega_t^n / n!$. We observe that Perelman's \mathcal{W} -functional [Per] satisfies the identity $\mathcal{W}(g_t, \Omega_t) = 2 \int_X f_t \Omega_t$. The monotony of this quantity along the Kähler-Ricci flow was discovered in [Pal1]. The inequality

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{W}(g_t, \Omega_t) \leq 0, \quad (1)$$

follows immediately from the existence of an identification map $\omega_t \longleftrightarrow J_t \equiv$ complex structure compatible with ω (see lemma 30 in [Pal2]), from the diffeomorphism invariance of \mathcal{W} and from author's solution of the variational stability problem for Kähler-Ricci solitons [Pal2, Pal3]. These facts imply also that the equality hold in (1) if and only if $\dot{\varphi}_0 = \psi + \bar{\psi}$ with $(\nabla_{g_0} \psi)_J^{1,0}$ holomorphic vector field (See proposition 2 and lemma 22 in [Pal2]).

We provide now an independent proof of the inequality (1) and of the identification of the kernel of the left hand side. The first variation of the quantity $\mathcal{W}_t := \mathcal{W}(g_t, \Omega_t)$ follows from a computation quite similar to one given in section 4 of [Pal1]. We include it here for readers convenience. We introduce the Ω -divergence operator acting on vector fields ξ as

$$\text{div}^\Omega \xi := \frac{d(\xi \lrcorner \Omega)}{\Omega}.$$

We define the weighted real Laplacian $\Delta_g^\Omega u := -\text{div}^\Omega \nabla_g u$, acting on functions u . We can assume without loss of generality that the potential φ_t is normalized in a way that $\int_X \dot{\varphi}_t \Omega_t \equiv 0$. (Indeed we can replace the potential φ_t with $\tilde{\varphi}_t = \varphi_t - \int_0^t ds \int_X \dot{\varphi}_s \Omega_s$). We consider now the function

$$\dot{\Omega}_t^* := \dot{\Omega}_t / \Omega_t = -\frac{1}{2} \Delta_{g_t} \dot{\varphi}_t - \dot{f}_t. \quad (2)$$

Time deriving the normalizing condition $\int_X \Omega_t \equiv 1$ we obtain

$$\int_X \dot{\Omega}_t^* \Omega_t \equiv 0. \quad (3)$$

Time deriving the identity $\omega_t = \text{Ric}(\Omega_t)$ we obtain $i\partial\bar{\partial}\dot{\varphi}_t = -i\partial\bar{\partial}\dot{\Omega}_t^*$. This combined with the normalization of φ_t and with (3) implies

$$\dot{\varphi}_t = -\dot{\Omega}_t^*. \quad (4)$$

Using (4) and (2) we obtain

$$2\dot{f}_t = -\Delta_{g_t}\dot{\varphi}_t + 2\dot{\varphi}_t. \quad (5)$$

We deduce the identities

$$\begin{aligned} \dot{\mathcal{W}}_t &= 2 \int_X \dot{f}_t \Omega_t + 2 \int_X f_t \dot{\Omega}_t^* \Omega_t \\ &= - \int_X \Delta_{g_t} \dot{\varphi}_t - 2 \int_X f_t \dot{\varphi}_t \Omega_t \\ &= \int_X \left(\langle \nabla_{g_t} \dot{\varphi}_t, \nabla_{g_t} f_t \rangle_{g_t} - 2f_t \dot{\varphi}_t \right) \Omega_t \\ &= \int_X (\Delta_{g_t}^{\Omega_t} f_t - 2f_t) \dot{\varphi}_t \Omega_t \\ &= \int_X (\Delta_{g_t}^{\Omega_t} f_t - 2f_t + \mathcal{W}_t) \dot{\varphi}_t \Omega_t. \end{aligned}$$

We set $g := g_0$, $\Omega := \Omega_0$ and $f := f_0$. It is well known (see for example [Pal2]) that the Kähler-Ricci soliton condition is equivalent to the equation

$$\Delta_g^{\Omega} f - 2f + \mathcal{W}_0 = 0.$$

We deduce

$$\begin{aligned} \ddot{\mathcal{W}}_0 &= \int_X \frac{d}{dt} \Big|_{t=0} (\Delta_{g_t}^{\Omega_t} f_t - 2f_t + \mathcal{W}_t) \dot{\varphi}_0 \Omega \\ &= \int_X \frac{d}{dt} \Big|_{t=0} (\Delta_{g_t}^{\Omega_t} f_t - 2f_t) \dot{\varphi}_0 \Omega, \end{aligned}$$

thanks to the normalizing condition on φ_t . We notice now the following elementary identity obtained in [Pal2] section 3.2

$$\frac{d}{dt} \Delta_{g_t}^{\Omega_t} f_t = \text{div}^{\Omega_t} (\dot{g}_t^* \nabla_{g_t} f_t) - \left\langle \nabla_{g_t} \dot{\Omega}_t^*, \nabla_{g_t} f_t \right\rangle_{g_t} + \Delta_{g_t}^{\Omega_t} \dot{f}_t,$$

where $\dot{g}_t^* := g_t^{-1} \dot{g}_t = \omega_t^{-1} i\partial\bar{\partial}\dot{\varphi}_t = \partial_{T_{X,J}}^{g_t} \nabla_{g_t} \dot{\varphi}_t$. We infer

$$\begin{aligned} 2 \frac{d}{dt} \Delta_{g_t}^{\Omega_t} f_t &= 2 \text{div}^{\Omega_t} \left(\partial_{T_{X,J}}^{g_t} \nabla_{g_t} \dot{\varphi}_t \cdot \nabla_{g_t} f_t \right) + 2 \langle \nabla_{g_t} \dot{\varphi}_t, \nabla_{g_t} f_t \rangle_{g_t} \\ &+ \Delta_{g_t}^{\Omega_t} (2\dot{\varphi}_t - \Delta_{g_t} \dot{\varphi}_t), \end{aligned}$$

thanks to (4) and (5). Applying now the complex weighted Bochner type identity (13.3) in [Pal2] with $\Omega = \omega^n/n!$ we obtain

$$2\partial_{T_{X,J}}^{*g_t} \partial_{T_{X,J}}^{g_t} \nabla_{g_t} \dot{\varphi}_t = \nabla_{g_t} \Delta_{g_t} \dot{\varphi}_t,$$

and thus $2 \operatorname{div}^{\Omega_t} \partial_{T_{X,J}}^{*g_t} \partial_{T_{X,J}}^{g_t} \nabla_{g_t} \dot{\varphi}_t = -\Delta_{g_t}^{\Omega_t} \Delta_{g_t} \dot{\varphi}_t$. We deduce

$$2 \frac{d}{dt} \Delta_{g_t}^{\Omega_t} f_t = 2 \operatorname{div}^{\Omega_t} \partial_{T_{X,J}}^{*g_t, \Omega_t} \partial_{T_{X,J}}^{g_t} \nabla_{g_t} \dot{\varphi}_t + 2 \langle \nabla_{g_t} \dot{\varphi}_t, \nabla_{g_t} f_t \rangle_{g_t} + 2 \Delta_{g_t}^{\Omega_t} \dot{\varphi}_t, \quad (6)$$

where $\partial_{T_{X,J}}^{*g_t, \Omega_t}$ is the adjoint of $\partial_{T_{X,J}}^{g_t}$ with respect to the volume form Ω_t . The proof of the identity (13.3) in [Pal2] shows that at a Kähler-Ricci soliton point (J, ω) hold the identity

$$2 \operatorname{div}^{\Omega} \partial_{T_{X,J}}^{*g, \Omega} \partial_{T_{X,J}}^g \nabla_g = -(\Delta_g^{\Omega})^2 - (B_{g,J}^{\Omega})^2, \quad (7)$$

where $B_{g,J}^{\Omega} u := \operatorname{div}^{\Omega}(J \nabla_g u) = g(\nabla_g u, J \nabla_g f)$. We consider now the complex weighted Laplacian

$$\Delta_{g,J}^{\Omega} = \Delta_g^{\Omega} - i B_{g,J}^{\Omega}.$$

In the Kähler-Ricci soliton case hold the identity $[\Delta_g^{\Omega}, B_{g,J}^{\Omega}] = 0$, (see (15.6) in [Pal2]). This combined with (7) implies

$$2 \operatorname{div}^{\Omega} \partial_{T_{X,J}}^{*g, \Omega} \partial_{T_{X,J}}^g \nabla_g = -\Delta_{g,J}^{\Omega} \overline{\Delta_{g,J}^{\Omega}}.$$

Using this and the identities (6) and (5), we obtain

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} (\Delta_{g_t}^{\Omega_t} f_t - 2f_t) &= \left(2\Delta_g^{\Omega} + 2 - \frac{1}{2} \Delta_{g,J}^{\Omega} \overline{\Delta_{g,J}^{\Omega}} \right) \dot{\varphi}_0 \\ &= -\frac{1}{2} P_{g,J}^{\Omega} \dot{\varphi}_0, \end{aligned}$$

where $P_{g,J}^{\Omega} := (\Delta_{g,J}^{\Omega} - 2\mathbb{I})(\overline{\Delta_{g,J}^{\Omega} - 2\mathbb{I}})$ is a non-negative self-adjoint real elliptic operator with respect to the L_{Ω}^2 -hermitian product (see [Pal2, Pal3]). We deduce

$$\ddot{\mathcal{W}}_0 = -\frac{1}{2} \int_X P_{g,J}^{\Omega} \dot{\varphi}_0 \cdot \dot{\varphi}_0 \Omega \leq 0.$$

This is a particular case of Proposition 2 in [Pal2]. The equality hold if and only if $\dot{\varphi}_0 = \psi + \bar{\psi}$ with $(\nabla_{g_0} \psi)_J^{1,0}$ holomorphic vector field thanks to lemma 22 in [Pal2].

Remark 1 Notice that the stability result over the space of Kähler potentials does not allow to deduce the general solution in [Pal2, Pal3]. This is because in the non Kähler-Einstein case (it has been pointed out in [Pal2] that in this case the solution is trivial), the tangent space of the embedding of the space of Kähler potentials inside the space of complex structures compatible with ω is not orthogonal and it has positive dimensional intersection with the tangent space to the symplectic orbit. Furthermore the understanding of the orthogonal behavior of the endomorphism Hessian of \mathcal{W} in restriction to such spaces is out of reach without using the general solution in [Pal2, Pal3].

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References

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